Persamaan Diferensial → Persamaan yang melibatkan turunannya

↳ Jenis : - Biasa → Single-variable. (ODE)
- Parsial → Multi-variabel, melibatkan turunan parsial } Tipe
- Orde → Derajat tertinggi dari turunan dalam PD

$y''' + y' = 2 \Rightarrow$ Orde 3

- Linearitas : $y^{(n)}$ memiliki koefisien, koefisiennya hanya fungsi $f_n(x)$, tidak ada komponen lain dari y.
- Autonomous: $\frac{dy}{dt} = f(y)$

↳ Solusi : $y'' - y' + 10y = 0$ ← Contoh soal

↳ Explicit → $y = xe^x$

Implicit → $y - xe^x = 0$

Trivial → $y = 0$

Order → Number of parameter in the family of solutions.

IVP → Punya Particular solution.

Syarat Cukup adanya solusi untuk IVP : $\left.\begin{array}{l} y' = f(x,y) \\ y(x_0) = y_0 \end{array}\right\}$ Orde 1

- R : area dalam bidang xy, dengan

$\left.\begin{array}{l} a \leq x \leq b \\ c \leq y \leq d \end{array}\right\}$ mengandung (x_0, y_0)

Jika $f(x,y)$ dan $\frac{\partial f}{\partial y}$ kontinu di R,

$\Rightarrow \exists$ interval I_0 dalam $a \leq x \leq b$

Fungsi $y(x)$ ada dan terdefinisi di I_0, Solusi dari IVP

ODE - Order 1

- Separable variables

$$\frac{dy}{dx} = f(x,y)$$

↳ Bila bisa dibuat $M(x)dx + N(y)dy = 0$, disebut separable.

↳ Simply integrate both sides.

- Linear Equations

$$\frac{dy}{dt} = y.p(y) + C \rightarrow \text{constant}$$

↳ Treat as separable

$$\frac{dy}{dt} + p(t).y = g(t)$$

$$y' + p(t).y = g(t)$$

Let: $\mu(t) = e^{\int p(t)dt} \Rightarrow \mu'(t) = p(t).\mu(t)$

$$\frac{d\mu}{dt} \times \frac{1}{\mu(t)} = p(t)$$

$$\mu(t)^{-1}.d\mu = p(t)\,dt$$

$$\ln|\mu(t)| + C_1 = \int p(t)\,dt$$

$$\mu(t) = e^{\int p(t).dt} > 0$$

$$\mu(t).(y' + p(t)y) = \mu(t).g(t)$$

$$\mu(t).y' + \underbrace{p(t)\mu(t)}.y = \mu(t).g(t)$$

$$\mu(t).y' + \mu'(t).y = \mu(t).g(t)$$

$$\frac{d}{dt}[\mu(t).y] = \mu(t).g(t)$$

$$d[\mu(t).y] = \mu(t).g(t).dt \quad \text{(Now let's integrate)}$$

$$\mu(t).y = \int \mu(t).g(t)\,dt + C_1$$

$$\boxed{y = \frac{\int \mu(t).g(t).dt + C_1}{\mu(t)} \;;\; \mu(t) = e^{\int p(t)\,dt}}$$

ODE - Order 1

- Exact Equations

Benduk ODE : $M(x,y) + N(x,y)y' = 0$

Jika $M_y(x,y) = N_x(x,y) \Rightarrow$ ODE is exact.

$$\frac{\partial M}{\partial y} \qquad \frac{\partial N}{\partial x}$$

Didapatkan :

$\psi_x(x,y) = M(x,y)$

$\psi(x,y) = \int M(x,y)\,dx$

$= F(x,y) + C(y) \leftarrow$ Komponen y parsial

$N(x,y) = \psi_y(x,y)$

$N(x,y) = F_y(x,y) + C'(y)$

$C'(y) = N(x,y) - F_y(x,y)$

$C(y) = \int N(x,y) - F_y(x,y)\,dy$

Solusi :

$y = F(x,y) + C(y) \leftarrow$ Implisit

- Integrating Factors for Almost Exact OE

Almost exact equation is when

$M_y(x,y) = N_x(x,y).(\text{something})$.

$M(x,y)\,dx + N(x,y)\,dy = 0 \leftarrow$ Almost exact, Find $\mu(x)$ or $\mu(y)$ so

$\mu(x)(M(x,y)\,dx + N(x,y)\,dy) = 0 \leftarrow$ Exact Equation

Special Integrating factor

$$\mu(x).\frac{\partial M}{\partial y} = \mu(x).\frac{\partial N}{\partial x} + N.\mu'(x)$$

$$\mu(x)\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial y}\right) = N.\mu'(x)$$

$$\int \frac{\mu'(x)}{\mu(x)} dx = \int \frac{\frac{\partial M}{dy} - \frac{\partial N}{\partial x}}{N} \partial x$$

$$\int \frac{1}{\mu(x)} d(\mu(x)) = \int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \partial x$$

$$\ln|\mu(x)| = \int \frac{M_y - N_x}{N} dx$$

$$\boxed{\begin{aligned} \mu(x) &= e^{\int \frac{M_y - N_x}{N} dx} \\ \mu(y) &= e^{\int \frac{N_x - M_y}{M} dy} \end{aligned}}$$

- Homogeneous Equation, ga ada konstan

$$\frac{\partial y}{\partial x} = f(x,y) = G(y/x)$$

Misalkan $\boxed{V = \frac{y}{x}}$ $\Rightarrow$ $y = Vx$

$dy = V\,dx + x\,dV$ $\nearrow$ $\frac{dy}{dx} = V + x\frac{dV}{dx}$

$\hookrightarrow V + x.\frac{dV}{dx} = G(V)$

$$x.\frac{dV}{dx} = G(V) - V$$

$$\int \frac{1}{G(V)-V} dV = \int \frac{1}{x} dx$$

Separable
Solve, resubstitule to $V = \frac{y}{x}$

Bernoulli's : $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^{r}$ $\leftarrow$ any real

Substitute $\boxed{V = y^{1-r}}$

It will be linear after substilution,

Example:

- $y' + y = -xy^3$

$V = y^{-2}$

$\frac{dV}{dx} = -2y^{-3}\frac{dy}{dx}$

- $(y' + y) . -2y^{-3} = -xy^3 . -2y^{-3}$

$\hookrightarrow \frac{dV}{dx} - 2V = 2x$

Later, resubstitute $V = y^{1-r}$

Aplikasi: Rate of population, constant to number of population:

$$\frac{dP}{dt} = \alpha P$$

1. Find α?, Solve using separable equation $\frac{1}{P} dP = \alpha \, dt \Leftrightarrow \int \frac{1}{P} dP = \int \alpha dt$

$\ln|P| = \alpha t + c$

$P = e^{\alpha t + c} = e^{\alpha t} \cdot e^{c}$

$= C \cdot e^{\alpha t}$

P : Population at time t
α, C : constant
↳ constant of proportionality
t : the time t

2. Substitute for each known t, and P(t). find α.

Mixture, amount of something inside a solution.

$$\frac{dA}{dt} = rate_{in} - rate_{out} = \Delta in - \Delta out$$

A : massa zat yang ada pada waktu-t
V : Volume zat yang ada pada waktu-t
t : waktu

Biasanya dalam satuan gram/menit atau massa/waktu.

Newton's law of cooling:

$$\frac{dT}{dt} = k(T - T_m)$$

T : Temperature of an object
T_m : surrounding temperature
k : constant

Chemical Reaction $(A + B \to C)$

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X)$$

X : the amount of Chemical C formed at time t : time
due to A and B
α : amount of A at t=0
β : amount of B at t=0
k : constant

Linearly Independent : Kombinasi linear homogen hanya memiliki tepat satu solusi, yaitu solusi trivial

Wronskian Determinant : $W = y_1 y_2' - y_1' y_2$ tidak 0, pada titik t_0, $y(t_0) = y_0$, $y'(t_0) = y_1'$

Maka akan ada solusi untuk c_1, c_2 untuk $y = c_1 y_1 + c_2 y_2$

Untuk membuktikan bahwa y_1, y_2 membentuk fundamental set of solution:

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ untuk semua t. Misal $y_1 = e^{-3x}$, $y_2 = e^{4x}$; $y'' - y' - 12y = 0$

1 $y_1 = e^{-3x}$
$y_1' = -3e^{-3x}$
$y_1'' = 9e^{-3x}$

$y'' - y' - 12y = 0$
$9e^{-3x} + 3e^{-3x} - 12e^{-3x} = 0$ ✓

2 $W = (e^{-3x} \cdot 4e^{4x}) - (e^{4x} \cdot -3e^{-3x})$
$7e^{x} \neq 0$ dalam interval $\in (-\infty, \infty)$

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3. Conclude

4 Construct all solution of IVP: General: $D \neq 0$, bebas linear $ye^{r_1 t} + e^{r_2 t}$, $r_1 \neq r_2$

Cek apakah merupakan solusi dari persamaannya.

Persamaan Diferensial Orde 2

General form :

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \longrightarrow y'' + p(x)y' + q(x)y = h(x)$

Bentuk yang akan dibahas :

$C_2 y'' + C_1 y' + C_0 y = g(x)$ → Koefisien konstan

$C_2 x^2 y'' + C_1 x y' + C_0 y = g(x)$ → Cauchy-euler equation

Super position :
if $y_1, y_2, \ldots y_k$ are solutions $\in I$
$y(x) = C_1 y_1 + C_2 y_2 + \ldots + C_k y_k$ is solution too.

IVP: $y(x_0) = y_0$, $y'(x_0) = y_0'$

Kapan IVP ada solusi ?

Jika $a_2(x)$, $a_1(x)$, $a_0(x)$ dan $g(x)$ kontinyu pada suatu interval I, dan $a_2(x) \neq 0$ untuk setiap $x \in I$, untuk $x = x_0$ di interval I, solusi $y(x)$ untuk IVP ada dan tunggal.

The 2nd order linear DE : $y'' - y = 0$, $y_1(t) = e^t$, $y_2(t) = e^{-t}$.

$y_t = C_1 e^t + C_2 e^{-t}$ → Family of Solution

Bila diberi syarat IVP, kita bisa dapatkan particular solution tanpa c nya.

BVP vs IVP

BVP : $y(x_0) = y_0$; $y(x_1) = y_1$

BVP : Solusi melalui dua titik yang diberikan
IVP : Syarat diberikan di suatu titik saja.

BVP bisa ada, bisa engga solusinya

Non - Homogenous

Characteristic Equation:

Goal ⇒ Solve homogenous constant coefficients high order DE;

$ay'' + by' + cy = 0$

Asumsikan solusi dalam bentuk $y = e^{rt}$

$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$

$e^{rt}(ar^2 + br + c) = 0$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Jika $b^2 - 4ac > 0$ = god bless u $\quad y_1 = e^{r_1 t}$; $y_2 = e^{r_2 t}$; $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$b^2 - 4ac == 0$: $y_1 = e^{r_1 t}$; $y_2 = x.e^{r_2 t}$ (obtained using $w' - w = 0$)

$b^2 - 4ac < 0$: use cosinehyp. and sinh.

$b^2 - 4ac = 0 \Rightarrow y_1 = e^{r_1 t}$, $y_2 = ?$; $y_2 = u.y_1$ (Karena bebas linear, substitusi ke awal)

$y_2 = x.e^{r_1 t}$ //

$y^2 + 25y = 0$

$\hookrightarrow m^2 + 25 = 0$

$m_{1,2} = \pm\sqrt{-25} = \pm 5i$

$y_1 = e^{5ix}$

$y_2 = e^{-5ix}$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$2.\cos x = e^{ix} + e^{-ix}$

$2.\cos(5x) = y_1 + y_2$

$2i.\sin(5x) = y_1 - y_2$ $\quad +$

$2.\cos(5x) + 2i\sin(5x) = 2y_1$

$y_1 = \cos(5x) + \sin(5x)i$

$y_2 = \underbrace{\cos(5x)}_{a} - \underbrace{\sin(5x)}_{b}i$

$(m-1)(m+2)^2 = 0$

$m_1 = 1$

$m_2 = m_3 = -2$

$y = c_1 e^x + c_2 e^{-2x} + c_3 xe^{-2x}$

$y(t) = c_1 e^{\lambda t}.\cos\mu t + c_2 e^{\lambda t}\sin\mu t$

Let $r_1 = \lambda + i\mu$

$r_2 = \lambda - i\mu$

$e^{it} = \cos t + i\sin t$

$e^{i\mu t} = \cos\mu t + i\sin\mu t$

$y'' + 4y = 0$, let $y = e^{rt}$ $\nearrow$ $a \pm bi$, $\lambda \pm \mu i$

$r^2 + 4 = 0 \Leftrightarrow r = \pm 2i$

$\lambda = 0$, $\mu = 2$

$y(t) = e^{at}(c_1 \cos \mu t + c_2 . \sin \mu t)$

$y(t) = c_1 . \cos(2t) + c_2 \sin(2t)$

$m_1 = 5 \rightarrow y_1$

$\left.\begin{matrix} m_2 = 2+i \\ m_3 = 2-i \end{matrix}\right\} \begin{matrix} y_2 \\ y_3 \end{matrix}$

$y(x) = c_1 e^{5x} + e^{2x}(c_2 \cos x + c_3 \sin x)$

Theorem : for $y'' + p(x)y' + q(x)y = g(x)$

$\underbrace{Y_1(x)}_{\text{Solution for non-homo}} - \underbrace{Y_2(x)}_{} = \underbrace{c_1 y_1(x) + c_2 y_2(x)}_{\text{Solution for homogeneous}}$

$Y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x)}_{\text{Solve this}} + \underbrace{Y_2(x)}_{\text{guess this from table}}$.

$g(x)$	y_p
1	A
$5x+7$	$Ax+B$
$\vdots$	$\vdots$

1. Guess y_p.
2. Find y_p, y'_p, y''_p
3. Substitute to
4. Find the constants of y_p
5. Mirip PFD
6. $y = y_h + y_p$

Jika y_h punya akar yang sama dengan y_p coba Axe^x

Goal : Solve simple non-homogeneous High order DE with constant coefficients

Reduction of Order

↳ Goal: Given a solution to a DE, find the other.

$t^2 y'' + 3ty' - 8y = 0$

$y_1 = t^2$

Let $y_2 = v(t).y_1 = v(t).t^2 = t^2 v$

$y_2' = t^2.v' + v.2.t = t^2v' + 2tv$

$y_2'' = t^2.v'' + v'.2.t + 2v. + 2t.v'$

$= t^2v'' + 2v't + 2v + 2tv'$

$= t^2v'' + 4tv' + 2v$

Substitusi ke

$t^2(t^2v'' + 4tv' + 2v) + 3t(t^2v' + 2tv) + (t^2v).-8 = 0$

$t^4v'' + 4t^3v' + \cancel{2t^2v} + 3t^3v' + \cancel{6t^2v} + \cancel{(-8t^2v)} = 0$

$t^4v'' + 7t^3v' = 0$

$v'' + \frac{7v'}{t} = 0$

Let $w = v'$

$\frac{dw}{dt} = -\frac{7w}{t}$ ← Separable

$\int \frac{1}{w} dw = \int -\frac{7}{t} dt$

$w = e^{-7.\ln|t| + k_1}$

$w = k_2 e^{-7\ln|t|}$

$v' = k_2 t^{-7}$

$\int \frac{dv}{dt} dt = \int k_2 t^{-7} dt$

$v = -\frac{1}{6} k_2.t^{-6} + k_4$

We know

$y_2 = v.t^2$

$= \left(-\frac{1}{6} k_2 t^{-6} + k_4\right) t^2$

$= -\frac{k_2.t^{-4}}{6} \boxed{+ k_4 t^2}$

$= k_3 t^{-4} + k_4 t^2$ Set of solution sama dari t^2

Fundamental set of Solutions.

Variation of Parameters:

Goal ⇒ When method of Undetermined Coefficients doesn't work!

$g(x)$ ⇒ Non linear or weird trancendental Function!

Mirip reduction of parameter, assume: $y_1 u_1' + y_2 u_2' = 0$

y_h

$y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

$y_p' = y_1' u_1 + u_2 y_2'$

$y_p'' = u_1 y_1'' + u_2 y_2'' + u_2' y_2' + u_1' y_1'$

Solve: Asumsikan $y_p = u_1(x) \cdot y_1(x) + u_2(x) \cdot y_2(x)$ → $ay'' + by' + cy = f(x)$

Berlaku asumsi juga: $y_1 u_1' + y_2 u_2' = 0$

$y_1' u_1' + y_2' u_2' = f(x)/a$ → System of linear equation, solve for u_1', u_2', then integrate.

Solusi: $u_1' = \dfrac{W_1}{W} = \dfrac{-y_2 f(x)}{W}$ $\qquad W = y_1 y_2' - y_1' y_2$

$u_2' = \dfrac{W_2}{W} = \dfrac{y_1 f(x)}{W}$

Cauchy - Euler DE

$ax^2 y'' + bxy' + cy = 0$

Asumsikan Solusi $y = x^r$

$x^r(ar^2 - ar + br + c) = 0$

$ar^2 - ar + br + c = 0$

$ar^2 + (-a+b)r + c = 0$

Case 1, $r_1 \neq r_2$

$y_1 = x^{r_1}$

$y_2 = x^{r_2}$

Case 2, x^{r_1}, $x^{r_1} \ln(x)$ ← reduction of order

Case 3, $r = \alpha \pm \beta i$

use: $x^{\alpha \pm \beta i}$

$x^{\alpha} \cdot \cos(\beta \ln x)$, $x^{\alpha} \sin(\beta \ln x)$

Shifted $(x-x_0)^2 y'' + \alpha (x-x_0) y' + \beta y = 0$

$y(x) = c_1 |x-x_0|^{m_1} + c_2 |x-x_0|^{m_2}$

⇒ $(c_1 + c_2 \ln|x-x_0|)|x-x_0|^{m_1}$

⇒ $c_1 |x-x_0|^{\lambda} \cos(\mu \ln|x-x_0|) + c_2 |x-x_0|^{\lambda} \sin(\mu \ln|x-x_0|)$ } $m = \lambda \pm i\mu$

plus

Reduction of order: $\ln^2(x)$

Modelling : Pegas.

W : berat dalam pounds

$m = \frac{W}{32}$ (massa dalam slug)

s : c. inch : $\frac{c}{12}$ ft (perpanjangan dalam feet)

$k = \frac{W}{s}$: (konstanta pegas dalam lbs/ft)

x = displacement dalam ft

$x(t) \Rightarrow$ fungsi displacement terhadap waktu, (sekon) atau terhadap satuan waktu lain

Equilibrium $\Rightarrow x(0)$.

negatif $\Rightarrow$ di atas / naik

positif $\Rightarrow$ turun / di bawah equilibrium.

Hukum hooke :

$$m x'' + \underbrace{\beta x'}_{\text{saat damped}} + k x = 0$$

Power Series Solution

Misalkan $y \Rightarrow \sum_{n=0}^{\infty} a_n x^n$

$$y' \rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' \rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) \cdot a_{n+2} x^n$$

Bila dari luar ada dikalikan sesuatu ($P(x)$, $Q(x)$ atau $R(x)$ kalikan ke series)

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

Maka basisnya sesuaikan.

Akan didapatkan a_n, $n \in \{0, 1, 2, \ldots\}$

Akan ada variabel bebas, berupa basis, biasanya a_0 dan a_1.

Laplace Transform: IVP $\rightarrow$ Laplace $\rightarrow$ Inverse Laplace $\rightarrow$ Solution

$F(s) \Rightarrow$ fungsi laplace di s-space

$f(t) \Rightarrow$ fungsi di t-space.

$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$ $\Leftarrow$ exist saat ini finite.

Laplace itu linear transformation.

Contoh: IVP $y(0) = 2$, $y'(0) = 3$, $y'' + 5y' + 6y = 0$

$$\mathcal{L}\{y'' + 5y' + 6y\} = \mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{0\} = \{0\}$$

$$\mathcal{L} \quad : s^2 \mathcal{L}\{y\} - s.y(0) - y'(0) + 5\, s\mathcal{L}\{y\} - y(0) + 6\mathcal{L}\{y\} = 0$$

Tulis $Y(s) = L\{y\}$

$$(s^2 + 5s + 6)Y(s) - (s+5)\,y(0) - y'(0) = 0$$

$$(s^2 + 5s + 6).Y(s) \,.\, -2(s+5) - 3 = 0$$

$$Y(s) = \frac{2s + 13}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$$

$A + B = 2$
$2A + 3B = 13$
$B = 9$, $A = -7$

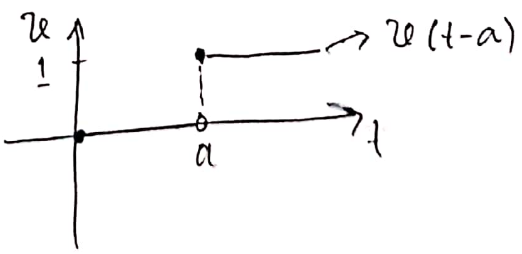
$$= \frac{-7}{s+3} + \frac{9}{s+2}$$

Cari $\mathcal{L}^{-1}\{Y(s)\}$ $\longrightarrow$ $\hookrightarrow -7.e^{-3t} + 9e^{2t}$

Translation: $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

Unit Step / Heaviside Function

$$u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$ $\Rightarrow$ Ditulis juga $u_a(t)$

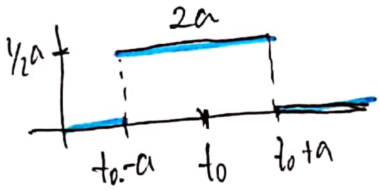


$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as} F(s)$$

$$\mathcal{L}\{g(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}$$

Unit Impulse Function

$$\delta_u(t-t_0) = \begin{cases} 0 & 0 \leq t < t_0 - a \\ \frac{1}{2a} & t_0 - a \leq t < t_0 + a \\ 0 & t \geq t_0 + a \end{cases}$$



$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$\delta(t-t_0) = \lim_{a \to 0} \delta_a(t-t_0)$$

$$\delta(t-t_0) = \begin{cases} \infty & t = t_0 \\ 0 & t \neq t_0 \end{cases}$$

$$\int_0^{\infty} \delta(t-t_0)\, dt = 1$$